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BRUNNIAN BRAIDS AND LIE ALGEBRAS

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Abstract. Brunnian braids have interesting relations with homotopy groups of spheres. In this work, we study the graded Lie algebra of the descending central series related to Brunnian subgroup of the pure braid group. A presentation of this Lie algebra is obtained.

1. Introduction

The pure braid group P_n (of a disc) can be given by the following presentation:
generators: $a_{i,j}$, $1 \leq i < j \leq n$,
the defining relations (*Burau relations* ([5], [20])):

$$
\begin{cases}
a_{i,j}a_{k,l} = a_{k,l}a_{i,j} \text{ for } i<j<k<l \text{ and } i<k<l<j, \\
a_{i,j}a_{i,k}a_{j,k} = a_{i,k}a_{j,k}a_{i,j} \text{ for } i<j<k, \\
a_{i,k}a_{j,k}a_{i,j} = a_{j,k}a_{i,j}a_{i,k} \text{ for } i<j<k, \\
a_{i,k}a_{j,k}a_{j,l}a_{j,k}^{-1} = a_{j,k}a_{j,l}a_{j,k}^{-1}a_{i,k} \text{ for } i<j<k<l.
\end{cases}
\tag{1.1}
$$

A geometric braid is called Brunnian if (1) it is a pure braid and (2) it becomes trival braid by removing any of its strands. Since the composition of any two Brunnian braids is still Brunnian, the set of Brunnian braids is a subgroup of the braid group which is denoted by Brun_n. By a direct geometric observation Brun_n is the normal subgroup of P_n, it is generated by the iterated commutators

$$[[[a_{1,2}, a_{i_2,3}], a_{i_3,4}], \ldots, a_{i_{n-1},n}]$$

for $1 \leq i_t \leq t$ and $2 \leq t \leq n-1$, where the commutator $[a,b]$ is defined by $[a,b] = a^{-1}b^{-1}ab$ for $a, b \in G$ [13].

Brunnian braids have connections with homotopy theory as described in [2], [14] and [1].

We remind that for a group G the descending central series

$$G = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_i \geq \Gamma_{i+1} \geq \ldots.$$

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is defined by the formulae

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$

$$L_i(G) = \Gamma_i(G)/\Gamma_{i+1}(G).$$

The descending central series and the associated Lie algebras of the pure braid groups have been studied in particular in the works [6, 8, 11, 12, 19]. It is also an ingredient in the study of Vassiliev invariants of braids. The associated graded algebra of the Vassiliev filtration for pure braid group ring coincides with the associated algebra of the filtration by the powers of augmentation ideal of the group ring of pure braids. The latter by the Quillen's theorem [16] is connected with the universal enveloping algebra of the associated Lie algebra of the descending central series of the pure braid group.

In this work, we consider the restriction $\{\Gamma_q(P_n) \cap \mathrm{Brun}_n\}$ of the descending central series of P_n to Brun_n. This gives a relative Lie algebra

$$\mathrm{L^P(Brun_n)} = \bigoplus_{q=1}^{\infty} (\Gamma_q(P_n) \cap \mathrm{Brun}_n)/(\Gamma_{q+1}(P_n) \cap \mathrm{Brun}_n), \tag{1.2}$$

which is a two-sided Lie ideal of $L(P_n)$. The purpose of this article is to study the Lie algebra $\mathrm{L^P(Brun_n)}$.

We remark that the group Brun_n is a free group of infinite rank for $n \geq 4$ and so the associated Lie algebra $L(\mathrm{Brun}_n)$ is an infinitely generated free Lie algebra for $n \geq 4$. The relative Lie algebra $\mathrm{L^P(Brun_n)}$ has better features, in particular it is of finite type (in graded sense).

The main aim of the paper is to look at Brunnian braids at the level of Lie algebras. Proposotions 2.2, 2.3 and 2.4 as well as some subsequent statements are the Lie algebra analogues of the corresponding facts for Brunnian groups.

2. Lie algebra $\mathrm{L^P(Brun_n)}$

A presentation of the Lie algebra $L(P_n)$ for the pure braid group can be described as follows [12]. It is the quotient of the free Lie algebra $L[A_{i,j} | \, 1 \leq i < j \leq n]$ generated by elements $a_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

$$
\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases} \tag{2.1}
$$

Where $A_{i,j}$ is the projection of the $a_{i,j}$ to $L(P_n)$.

Let G be a group with filtration w (in the sense of Serre [17, p. 7]). The fact that $\mathrm{L^P(Brun_n)}$ as defined in (1.2) is a Lie algebra is a corollary of the following evident statement.

Proposition 2.1. *For any subgroup H of G the restriction on H of filtration w defines a filtration on H.* □

We define a filtration w_B on Brun_n by the formula:

$$
w_B(b) = \inf\{p \mid b \in \Gamma_p\}.
$$

Proposition 2.2. $\mathrm{L^P(Brun_n)}$ *is a Lie algebra defined by the filtration w_B, it is a two-sided Lie ideal in $L(P_n)$.*

Proof. The last statement follows from the fact that Brun_n is a normal subgroup of P_n. □

We call $\mathrm{L^P(Brun_n)}$ *relative Lie algebra associated with Brunnian subgroup* of the pure braid group.

The removing-strand operation on braids induces an operation

$$
d_k \colon L(P_n) \longrightarrow L(P_{n-1})
$$

formulated by

$$
d_k(A_{i,j}) = \begin{cases} A_{i,j} & \text{if} \quad i < j < k \\ 0 & \text{if} \quad k = j \\ A_{i,j-1} & \text{if} \quad i < k < j \\ 0 & \text{if} \quad k = i \\ A_{i-1,j-1} & \text{if} \quad k < i < j. \end{cases} \tag{2.2}
$$

A sequence of sets $S = \{S_n\}_{n\geq 0}$ is called a bi-Δ-set if there are faces $d_j : S_n \to S_{n-1}$ and co-faces $d^j : S_{n-1} \to S_n$ for $0 \leq j \leq n$ such that the following identies hold:

(1). $d_j d_i = d_i d_{j+1}$ for $j \geq i$;
(2). $d^j d^i = d^{i+1} d^j$ for $j \leq i$;
(3). $d_j d^i = \begin{cases} d^{i-1} d_j & \text{if} \quad j < i, \\ \mathrm{id} & \text{if} \quad j = i, \\ d^i d_{j-1} & \text{if} \quad j > i. \end{cases}$

In other words, S is a bi-Δ- and co-Δ-set such that relation (1)-(3) holds. Moreover a sequence of groups $\mathcal{G}$ is called a bi-Δ-group if $\mathcal{G}$ is a bi-Δ-set such that all faces and co-faces are group homomorphism.

Let $\mathbb{P}_n = P_{n+1}$. According to [23, Example 1.2.8], the sequence of groups $\mathbb{P} = \{\mathbb{P}_n\}_{n\geq 0}$ with faces relabeled as $\{\mathbb{d}_0, \mathbb{d}_1, \ldots\}$ and co-faces relabeled as $\{\mathbb{d}^0, \mathbb{d}^1, \ldots\}$ forms a bi-Δ-group structure. Where the face operation $\mathbb{d}_i : \mathbb{P}_n \to \mathbb{P}_{n-1} = d_{i+1} : P_{n+1} \to P_n$ is obtained by deleting the $i+1$st string, the co-face operation $\mathbb{d}^i : \mathbb{P}_n \to \mathbb{P}_{n+1}$ is obtained by adding a trivial $i+1$st string in front of the other strings $(i = 0, 1, 2, \cdots, n)$.

Proposition 2.3. *The relative Lie algebra* $\mathrm{L^P(Brun_n)}$ *is the Lie subalgebra* $\bigcap_{i=1}^{n} \ker(d_i : L(P_n) \to L(P_{n-1}))$.

Proof. The assertion follows from [23, Proposition 1.2.10]. □

Our next step is to determine a set of generators for the Lie algebra $\mathrm{L^P(Brun_n)}$. The following fact is a Lie algebra analogue of the theorem proved by A. A. Markov [15] for the pure braid group. Also it follows from Theorem 3.1 in [7] or Lemma 3.1.1 in [10].

Proposition 2.4. *The kernel of the homomorphism* $d_n : L(P_n) \to L(P_{n-1})$ *is a free Lie algebra, generated by the free generators* $A_{i,n}$*, for* $1 \leq i \leq n-1$.

$$\mathrm{Ker}(d_n : L(P_n) \to L(P_{n-1})) = L[A_{1,n}, \ldots, A_{n-1,n}].$$

□

For a set Z, let $L[Z]$ denote the free Lie algebra freely generated by Z. Let X and Y be nonempty (possibly infinite) sets with $X \cap Y = \emptyset$, $X \cup Y = Z$. We are interested to study the kernel of Lie homomorphism

$$\pi \colon L[Z] \longrightarrow L[Y]$$

π such that $\pi(x) = 0$ for $x \in X$ and $\pi(y) = y$ for $y \in Y$. The following lemma is not new: for the case of Lie algebras over a field and when X consists of one element this is Lemma 2.6.2 in [3]. For completeness of our exposition we are giving our proof here.

Lemma 2.5. *The kernel of* π *is a free Lie algebra, generated by the following family of free generators:*

$$x, [\cdots [x, y_1], \ldots, y_t] \tag{2.3}$$

for $x \in X, y_i \in Y$ *for* $1 \leq i \leq t$.

Proof. Observe that the kernel $\mathrm{Ker}(\pi)$ is the two-sided ideal generated by the elements $x \in X$. Let us prove that it is generated as a Lie algebra by the elements (2.3). We prove this by induction on the length of monomials M, sums of which give the ideal. For the length 1 and 2 one can see this directly. Let the length of M be at least 3: $M = [A, B]$ such that A contains some $x \in X$. We may assume that the length of A is at least 2. If not, then $A = x$ for some $x \in X$ and $B = [B_1, B_2]$ and $M = [x, [B_1, B_2]] = [[B_2, x], B_1] + [[x, B_1], B_2]$ and it is reduced to the case when in M element A contains some $x \in X$ with its length at least 2. Since $A \in \mathrm{Ker}(\pi)$ with its length strictly less than that of M, it is a linear combination of the products $[A_1, A_2]$ of the generators (2.3) with both A_1 and A_2 containing some (possibly different) element(s) in X by induction. Consider the equality

$$[[A_1, A_2], B] = -[[A_2, B], A_1] - [[B, A_1], A_2].$$

The elements $[A_2, B]$ and $[B, A_1]$ have length strictly less than that of M and by induction they are given by linear combinations of products of the generators (2.3). Thus $M = [A, B]$ is a linear combination of products of the generators (2.3).

Let us prove now that the elements (2.3) freely generate our ideal $\mathrm{Ker}(\pi)$. Let us define a free Lie algebra (over $\mathbb{Z}$) that is freely generated by formal elements $\{C(x),\ C(x, y_1, y_2, \ldots, y_t)\}$, $x \in X,\ y_1, \ldots, y_t \in Y$ with $t \geq 1$, which are in one-to-one correspondence with the elements (2.3)

$$F = L[C(x),\ C(x, y_1, y_2, \ldots, y_t)\ |x \in X,\ y_1, \ldots, y_t \in Y,\ t \geq 1].$$

Let us define an action of the free Lie algebra $L[X \sqcup Y]$ on F by the formulae which mimic the action of $L[X \sqcup Y]$ on the elements (2.3):
(2.4)

$$\begin{cases}[C(x, y_1, y_2, \ldots, y_t), y] = C(x, y_1, y_2, \ldots, y_t, y), y \in Y\\ [C(x, y_1, y_2, \ldots, y_t), x'] = [C(x, y_1, y_2, \ldots, y_t), C(x')], x' \in X,\end{cases}$$

where, for $t = 0$, $C(x, y_1, y_2, \ldots, y_t) = C(x)$. Let us denote the generators (2.3) of our ideal by $B(x)$, $B(x, y_1, y_2, \ldots, y_t)$, $x \in X$, $y_1, \ldots, y_t \in Y$ with $t \geq 1$. We claim that the element $B(x, y_1, y_2, \ldots, y_t)$ acts on F the same way as the inner derivation by $C(x, y_1, y_2, \ldots, y_t)$:

$$\Big\{[C(x', y'_1, \ldots, y'_{t'}), B(x, y_1, \ldots, y_t)] = [C(x', y'_1, \ldots, y'_{t'}), C(x, y_1, \ldots, y_t)].$$

The proof is by induction on the length t of $B(x, y_1, \ldots, y_t)$. For the length $t = 0$ it follows from the definition of the action. Let it be proved for the lengths less than t with $t > 0$. Let $D = C(x', y'_1, \ldots, y'_{t'})$, $C =$

$C(x, y_1, \ldots, y_t)$, $B' = B(x, y_1, \ldots, y_{t-1})$ and $C' = C(x, y_1, \ldots, y_{t-1})$.

$$\begin{aligned}
[D, B(x, y_1, \ldots, y_t)] &= [D, [B', y_t]] \\
&= [[D, B'], y_t] - [[D, y_t], B'] \\
&= [[D, C'], y_t] - [C(x', y'_1, \ldots, y'_{t'}, y_t), B'] \\
&\qquad \text{(by induction)} \\
&= [[D, C'], y_t] - [C(x', y'_1, \ldots, y'_{t'}, y_t), C'] \\
&\qquad \text{(by induction)} \\
&= [[D, C'], y_t] - [[D, y_t], C'] \\
&= [D, [C', y_t]] \\
&= [D, C].
\end{aligned}$$

The induction is finished.

Let $D(F)$ be the Lie algebra of all derivations of the algebra F, homomorphism $\chi : F \to D(F)$ is defined by inner derivations. We define homomorphism $\phi : F \to L[X \sqcup Y]$ by the formulae

$$\phi(C(x)) = B(x) = x \quad \text{and}$$

$$\phi(C(x, y_1, \ldots, y_t)) = B(x, y_1, \ldots, y_t) = [\cdots [x, y_1], \ldots, y_t]$$

and homomorphism $\delta : L[X \sqcup Y] \to D(F)$ is defined by the action (2.4). There is a commutative diagram:

$$\begin{tikzcd}
F \arrow[rr, "\phi"] \arrow[dr, "\chi"'] & & L[X \sqcup Y] \arrow[dl, "\delta"] \\
& D(F). &
\end{tikzcd}$$

The homomorphism χ is a monomorphism as free Lie algebras with more than 2 generators have trivial center [4, Exercice 3), §3, p. 79]. So ϕ is also a monomorphism and hence it is an isomorphism on the ideal, generated by $B(x, y_1, \ldots, y_t)$.

□

Proposition 2.6. *The intersection of the kernels of the homomorphisms d_n and d_k, $k \neq n$, is a free Lie algebra, generated by the following infinite family of free generators:*

$$A_{k,n}, [\cdots [A_{k,n}, A_{j_1,n}], \ldots, A_{j_m,n}] \tag{2.5}$$

for $j_i \neq k, n$; $j_i \leq n-1$; $i \leq m$; $m \geq 1$:

(2.6) $\operatorname{Ker}(d_n) \cap \operatorname{Ker}(d_k) =$

$L[A_{k,n},\ [\cdots [A_{k,n}, A_{j_1,n}], \ldots, A_{j_m,n}] \mid j_i \neq k, n;\ j_i \leq n{-}1, i \leq m;\ m \geq 1].$

Proof. Let us suppose for simplicity that $k = n - 1$ and denote $A_{i,n}$ by B_i. Then the algebra from Proposition 2.4 is the following free Lie algebra $L[B_1, \dots, B_k]$, and the homomorphism d_k can be expressed by the formulae

$$\begin{cases} B_1 \mapsto B_1, \\ \quad \cdots, \\ B_{k-1} \mapsto B_{k-1}, \\ B_k \mapsto 0. \end{cases}$$

The assertion follows from Lemma 2.5. □

Another set of free generators of $\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$ can be obtained using Hall bases [4], [9]. We remind the definition. We suppose that all Lie monomials on $B_1, \dots, B_k$ are ordered lexicographically.

Lie monomials $B_1, \dots, B_k$ are the *standard* monomials of degree 1. If we have defined standard monomials of degrees $1, \dots, n - 1$, then $[u, v]$ is a *standard* monomial if both of the following conditions hold:

(1) u and v are standard monomials and $u > v$.

(2) If $u = [x, y]$ is the form of the standard monomial u, then $v \geq y$.

Standard monomials form the *Hall basis* of a free Lie algebra (also over $\mathbb{Z}$). Examples of standard monomials are the products of the type:

$$[\cdots[B_{j_1}, B_{j_2}], B_{j_3}], \dots, B_{j_t}], \; j_1 > j_2 \leq j_3 \leq \cdots \leq j_t. \tag{2.7}$$

Proposition 2.7. *The intersection* $\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_k)$*,* $k \neq n$*, is a free Lie algebra, generated by the standard monomials on* $A_{i,n}$ *where the letter* $A_{k,n}$ *has only one enter. In other words the free generators are standard monomials which are products of monomials of type (2.7) where only one such monomial contains one copy of* $A_{k,n}$*.*

Proof. We apply the procedure of constructing of a set of free generators for a sub Lie algebra which was used by Shirshov [18] and Witt [21] in their proofs that a Lie subalgebra of a free Lie algebra is free. □

Lemma 2.5 is useful for having an algorithm to recursively determine a set of free generators for $\mathrm{L^P(Brun_n)}$. A *Lie monomial* W on the letters $A_{1,n}, A_{2,n}, \dots, A_{n-1,n}$ means $W = A_{i,n}$ for some $1 \leq i \leq n-1$ or a Lie bracket $W = [A_{j_1,n}, A_{j_2,n}, \dots, A_{j_t,n}]$ under any possible bracket arrangements with entries on the letters $A_{i,n}$'s.

Definition 2.8. We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_{1,n}, A_{2,n}, \dots, A_{n-1,n}\}$.

2) Suppose that $\mathcal{K}(n)_{k+1}$ is defined as a subset of Lie monomials on the letters

$$A_{1,n}, A_{2,n}, \dots, A_{n-1,n}$$

with $k < n$. Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries}\}.$$

3) Define

$$\mathcal{K}(n)_k = \{W' \text{ and } [\cdots[[W', W_1], W_2], \dots, W_t]\}$$

for $W' \in \mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k$ and $W_1, W_2, \dots, W_t \in \mathcal{A}_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_{1,n}, A_{2,n}, \dots, A_{n-1,n}$.

Example 2.9. Let $n = 3$. The set $\mathcal{K}(3)_1$ is constructed by the following steps:

1) $\mathcal{K}(3)_3 = \{A_{1,3}, A_{2,3}\}$.
2) $\mathcal{A}_2 = \{A_{1,2}\}$,

$$\mathcal{K}(3)_2 = \{A_{2,3}, [[A_{2,3}, A_{1,3}], \dots, A_{1,3}]\}.$$

3) $\mathcal{A}_1 = \{A_{2,3}\}$,

$$\mathcal{K}(3)_1 = \{[\cdots[A_{2,3}, A_{1,3}], \dots, A_{1,3}], A_{2,3}], \dots, A_{2,3}]\}.$$

Remark 2.10. *All elements of $\mathcal{K}(3)_1$ under the canonical inclusion $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n}) \hookrightarrow L(P_3)$ are mapped to the elements (not all) of a Hall basis for the free Lie subalgebra of of $L(P_3)$ generated by $A_{1,3}$ and $A_{2,3}$.*

Theorem 2.11. *The Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun_n})$ is a free Lie algebra generated by $\mathcal{K}(n)_1$ as a set of free generators.*

Proof. The assertion follows from the statement that $\mathcal{K}(n)_k$ is a set of free generators for

$$\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_{n-1}) \cap \cdots \cap \mathrm{Ker}(d_k)$$

for $1 \leq k \leq n$. We prove this statement by induction in reverse order. For $k = n$ it follows from Proposition 2.4. Suppose that the statement holds for $k+1$ with $k < n$. Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries}\}$$

and let $\mathcal{B}_k = \mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k$. By induction,

$$\mathrm{Ker}(d_n) \cap \mathrm{Ker}(d_{n-1}) \cap \cdots \cap \mathrm{Ker}(d_{k+1}) = L[\mathcal{K}(n)_{k+1}]$$

is a free Lie algebra freely generated by $\mathcal{K}(n)_{k+1}$. The Lie algebra generated by $\mathcal{A}_k$ is a Lie subalgebra of the Lie algebra freely generated by $A_{1,n}, \dots, A_{n-1,n}$. Thus the Lie homomorphism

$$\phi\colon L[\mathcal{A}_k] \longrightarrow L[A_{1,n}, \dots, A_{n-1,n}]$$

with $\phi(W) = W$ for $W \in \mathcal{A}_k$ is a monomorphism with its image given by the Lie subalgebra generated by $\mathcal{A}_k$.

Consider the homomorphism

$$d_k\colon L[A_{1,n}, \ldots, A_{n-1,n}] \longrightarrow L[A_{1,n-1}, \ldots, A_{n-2,n-1}]$$

given in formula (2.2). We show that the composite

$$d_k \circ \phi\colon L[\mathcal{A}_k] \longrightarrow L[A_{1,n-1}, \ldots, A_{n-2,n-1}].$$

is a monomorphism. By the definition of $\mathcal{A}_k$, the image $\phi(L[\mathcal{A}_k])$ is contained in the Lie subalgebra

$$L[A_{1,n}, \ldots, A_{k-1,n}, A_{k+1,n}, \ldots, A_{n-1,n}]$$

of $L[A_{1,n}, \ldots, A_{n-1,n}]$. Thus there is a commutative diagram of Lie algebras

$$\begin{tikzcd}
L[\mathcal{A}_k] \arrow[r, "\phi"] \arrow[d, "\phi'"] & L[A_{1,n}, \ldots, A_{n-1,n}] \arrow[d, "d_k"] \\
L[A_{1,n}, \ldots, A_{k-1,n}, A_{k+1,n}, \ldots, A_{n-1,n}] \arrow[ur, hook] \arrow[r, "d_k|"] & L[A_{1,n-1}, \ldots, A_{n-2,n-1}],
\end{tikzcd}$$

where ϕ' is defined by the same formula as ϕ. Since ϕ is a monomorphism, so is ϕ'. From the definition, the restriction

$$d_k|\colon L[A_{1,n}, \ldots, A_{k-1,n}, A_{k+1,n}, \ldots, A_{n-1,n}] \longrightarrow L[A_{1,n-1}, \ldots, A_{n-2,n-1}]$$

is an isomorphism. It follows that $d_k \circ \phi\colon L[\mathcal{A}_k] \to L[A_{1,n-1}, \ldots, A_{n-2,n-1}]$ is a monomorphism.

Observe that $d_k(W) = 0$ for $W \in \mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k$. There is a commutative diagram of Lie algebras

$$\begin{tikzcd}
L[\mathcal{K}(n)_{k+1}] \arrow[r, equal] \arrow[d, "\pi"] & \mathrm{Ker}(d_n) \cap \cdots \cap \mathrm{Ker}(d_{k+1}) \arrow[r, hook] \arrow[dr, "d_k|"] & L[A_{1,n}, \ldots, A_{n-1,n}] \arrow[d, "d_k"] \\
L[\mathcal{A}_k] \arrow[rr, hook, "d_k \circ \phi"] & & L[A_{1,n-1}, \ldots, A_{n-2,n-1}],
\end{tikzcd}$$

where $\pi(W) = 0$ for $W \in \mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k$ and $\pi(W) = W$ for $W \in \mathcal{A}_k$. It follows that

$$\mathrm{Ker}(d_n) \cap \cdots \cap \mathrm{Ker}(d_k) =$$

$$= \mathrm{Ker}(d_k|\colon \mathrm{Ker}(d_n) \cap \cdots \cap \mathrm{Ker}(d_{k+1}) \to L[A_{1,n-1}, \ldots, A_{n-2,n-1}])$$

is given by the kernel of

$$\pi\colon L[\mathcal{K}(n)_{k+1}] = L[(\mathcal{K}(n)_{k+1} \smallsetminus \mathcal{A}_k) \sqcup \mathcal{A}_k] \longrightarrow L[\mathcal{A}_k],$$

which is freely generated by $\mathcal{K}(n)_k$ by Lemma 2.5. This finishes the proof. $\square$

Example 2.12. Let $n = 4$. The set $\mathcal{K}(4)_1$ is constructed by the following steps:

1) $\mathcal{K}(4) = \{A_{1,4}, A_{2,4}, A_{3,4}\}$.
2) $\mathcal{A}_3 = \{A_{1,4}, A_{2,4}\}$,
$\mathcal{K}(4)_3 = \{[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}] \mid 1 \leq j_1, \ldots, j_t \leq 2,\ t \geq 0\}$,
where, for $t = 0$, $[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}] = A_{3,4}$.
3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}] \in \mathcal{K}(4)_3$. If W does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}]$ with $j_1 = j_2 = \cdots = j_t = 1$. Let
$$\mathrm{ad}^t(b)(a) = [[a, b], b, \ldots, b]$$
with t entries of b, where $\mathrm{ad}^0(b)(a) = a$. Then W does not contain $A_{2,4}$ if and only if
$$W = \mathrm{ad}^t(A_{1,4})(A_{3,4})$$
for $t \geq 0$. So $\mathcal{A}_2 = \{\mathrm{ad}^t(A_{1,4})(A_{3,4}), t \geq 0\}$. From the definition, $\mathcal{K}(4)_2$ is given by
$$[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}] \quad \text{and}$$
$$[[[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \ldots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})],$$
where $1 \leq j_1, \ldots, j_t \leq 2$ with at least one $j_i = 2$, $s_1, \ldots, s_q \geq 0$ and $q \geq 1$.
4) For constructing $\mathcal{K}(4)_1$, let W be an element of $\mathcal{K}(4)_2$,
$$W =$$
$$[[[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \ldots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})],$$
where, for $q = 0$, $W = [[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}]$. Then W does not contain $A_{1,4}$ if and only if $q = 0$ and $W = [[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}]$ with $j_1 = j_2 = \cdots = j_t = 2$, namely
$$W = \mathrm{ad}^t(A_{2,4})(A_{3,4})$$
for $t \geq 1$. So, $\mathcal{A}_1 = \{\mathrm{ad}^t(A_{2,4})(A_{3,4}), t \geq 1\}$. Thus $\mathcal{K}(4)_1$, which is a set of free generators for $\mathrm{L}^P(\mathrm{Brun}_4)$, is given by
$$W \text{ and } [[W, \mathrm{ad}^{l_1}(A_{2,4})(A_{3,4})], \ldots, \mathrm{ad}^{l_p}(A_{2,4})(A_{3,4})],$$
where $l_i \geq 1$ for $1 \leq i \leq p$ with $p \geq 1$ and
$$W =$$
$$[[[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \ldots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})]$$

is an element of $\mathcal{K}(4)_2$, so that each of $A_{2,4}$ and $A_{1,4}$ appears in W at least once. □

From the above example, one can see that the set $\mathcal{K}(n)_1$ is still complicated in the sense that its elements involve the iterated operations of normal Lie brackets from left to right $[\cdots[\ ,\ ],\ldots,\ ]$.

Question 2.13. *Determine a set of free generators for* $\mathrm{L^P(Brun_n)}$ *using normal Lie brackets from left to right.*

3. The symmetric Lie products of Lie ideals

Let L be a Lie algebra and $I_1,\ldots,I_n$ its ideals. We define the notion of the fat bracket sum and the symmetric bracket sum of ideals which is similar to the corresponding fat commutator product and symmetric commutator product in groups [1], [14]. Given a Lie algebra L, and a set of its ideals $I_1,\ldots,I_n,\ (l\geq 2)$, the fat bracket sum of these ideals is defined to be the Lie ideal of L generated by all of the commutators

$$\beta^t(a_{i_1},\ldots,a_{i_t}), \tag{3.1}$$

where

1) $1\leq i_s\leq n$;
2) $\{i_1,\ldots,i_t\}=\{1,\ldots,n\}$, so, each integer in $\{1,2,\cdots,n\}$ appears as at least one of the integers i_s;
3) $a_j\in I_j$;
4) β^t runs over all of the bracket arrangements of weight t (with $t\geq n$).

The symmetric bracket sum of these ideals is defined as

$$[[I_1,I_2],\ldots,I_l]_S:=\sum_{\sigma\in\Sigma_l}[[I_{\sigma(1)},I_{\sigma(2)}],\ldots,I_{\sigma(n)}],$$

where Σ_n is the symmetric group o n letters.

As in [1], [14] we can prove that the symmetric bracket sum of $I_1,\ldots,I_n,\ (n\geq 2)$ is the same as the fat bracket sum.

Theorem 3.1. *Let I_j be any Lie ideals of a Lie algebra L with $1\leq j\leq n$. Then*

$$[[I_1,I_2,\ldots,I_n]]=[[I_1,I_2],\ldots,I_n]_S.$$

To prove this theorem, we need the following lemmas.

Lemma 3.2. *Let L be a Lie algebra and let A,B,C be Lie ideals of L. Then any one of the Lie ideal $[A,[B,C]]$, $[[A,B],C]$ and $[[A,C],B]$ is a Lie ideal of the sum of the other two.* □

Lemma 3.3. *Let $I_1, \ldots, I_n$ be Lie ideals of L. Let $a_j \in I_j$ for $1 \leq j \leq n$. Then*

$$\beta^n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \in [[I_1, I_2], \ldots, I_n]_S$$

for any $\sigma \in \Sigma_n$ and any bracket arrangement β^n of weight n.

Proof. The proof is given by double induction. The first induction is on n. Clearly the assertion holds for $n = 1$. Suppose that the assertion holds for m with $m < n$. Given an element $\beta^n(a_{\sigma(1)}, \ldots a_{\sigma(n)})$ as in the statement of the lemma we have

$$\beta^n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = [\beta^p(a_{\sigma(1)}, \ldots, a_{\sigma(p)}), \beta^{n-p}(a_{\sigma(p+1)}, \ldots, a_{\sigma(n)})]$$

for some bracket arrangements β^p and β^{n-p} with $1 \leq p \leq n-1$. The second induction is on $q = n - p$. If $q = 1$, we have

$$\beta^{n-1}(a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}) \in [[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(n-1)}]_S$$

by the first induction and so

$$\begin{array}{rcl} \beta^n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) & = & [\beta^{n-1}(a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}), a_{\sigma(n)}] \\ & \in & [[[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(n-1)}]_S, I_{\sigma(n)}] \end{array}$$

with

$$\begin{array}{rl} & [[[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(n-1)}]_S, I_{\sigma(n)}] \\ = & \left[\sum_{\tau \in \Sigma_{n-1}} [[I_{\tau(\sigma(1))}, I_{\tau(\sigma(2))}], \ldots, I_{\tau(\sigma(n-1))}], I_{\sigma(n)}\right] \\ = & \sum_{\tau \in \Sigma_{n-1}} [[[I_{\tau(\sigma(1))}, I_{\tau(\sigma(2))}], \ldots, I_{\tau(\sigma(n-1))}], I_{\sigma(n)}] \\ \leq & [[I_1, I_2], \ldots, I_n]_S. \end{array}$$

Now suppose that the assertion holds for $q' = n - p < q$. By the first induction, we have

$$\beta^p(a_{\sigma(1)}, \ldots, a_{\sigma(p)}) \in [[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(p)}]_S$$

and

$$\beta^{n-p}(a_{\sigma(p+1)}, \ldots, a_{\sigma(n)}) \in [[I_{\sigma(p+1)}, I_{\sigma(p+2)}], \ldots, I_{\sigma(n)}]_S.$$

Thus

$$\beta^n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \in \left[[[I_{\sigma(1)}, I_{\sigma(2)}], \ldots, I_{\sigma(p)}]_S, [[I_{\sigma(p+1)}, I_{\sigma(p+2)}], \ldots, I_{\sigma(n)}]_S\right].$$

Then

$$\beta^n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \in \sum_{\substack{\tau \in \Sigma_p \\ \rho \in \Sigma_{n-p}}} \left[[[I_{\tau(\sigma(1))}, \ldots, I_{\tau(\sigma(p))}], [[I_{\rho(\sigma(p+1))}, \ldots, I_{\rho(\sigma(n))}]\right],$$

where Σ_{n-p} acts on $\{\sigma(p+1),\ldots,\sigma(n)\}$. By applying Jacobi identity, we have

$$\begin{aligned}
& \left[[[I_{\tau(\sigma(1))},\ldots,I_{\tau(\sigma(p))}],[[I_{\rho(\sigma(p+1))},\ldots,I_{\rho(\sigma(n))}]\right]\\
= & \left[[[I_{\tau(\sigma(1))},\ldots,I_{\tau(\sigma(p))}],\left[[[I_{\rho(\sigma(p+1))},\ldots,I_{\rho(\sigma(n-1))}],I_{\rho(\sigma(n))}\right]\right]\\
\leq & \left[\left[[[I_{\tau(\sigma(1))},\ldots,I_{\tau(\sigma(p))}],[[I_{\rho(\sigma(p+1))},\ldots,I_{\rho(\sigma(n-1))}]\right],I_{\rho(\sigma(n))}\right]\\
& +\left[\left[[[I_{\tau(\sigma(1))},\ldots,I_{\tau(\sigma(p))}],I_{\rho(\sigma(n))}\right],[[I_{\rho(\sigma(p+1))},\ldots,I_{\rho(\sigma(n-1))}]\right].
\end{aligned}$$

Note that $A=\left[\left[[[I_{\tau(\sigma(1))},\ldots,I_{\tau(\sigma(p))}],[[I_{\rho(\sigma(p+1))},\ldots,I_{\rho(\sigma(n-1))}]\right],I_{\rho(\sigma(n))}\right]$ is generated by the elements of the form

$$\left[\left[[[a'_{\tau(\sigma(1))},\ldots,a'_{\tau(\sigma(p))}],[[a'_{\rho(\sigma(p+1))},\ldots,a'_{\rho(\sigma(n-1))}]\right],a'_{\rho(\sigma(n))}\right]$$

with $a'_j\in I_j$. By the second induction in case when $q=1$, the above elements lie in $[[I_1,I_2],\ldots,I_n]_S$ and so

$$A\leq[[I_1,I_2],\ldots,I_n]_S.$$

Similarly, by the second induction hypothesis,

$$\left[\left[[[I_{\tau(\sigma(1))},\ldots,I_{\tau(\sigma(p))}],I_{\rho(\sigma(n))}\right],[[I_{\rho(\sigma(p+1))},\ldots,I_{\rho(\sigma(n-1))}]\right]$$

is a Lie ideal of $[[I_1,I_2],\ldots,I_n]_S$. It follows that

$$T\leq[[I_1,I_2],\ldots,I_n]_S$$

and so

$$\beta^n(a_{\sigma(1)},\ldots,a_{\sigma(n)})\in[[I_1,I_2],\ldots,I_n]_S.$$

Both the first and second inductions are finished, hence the result. $\square$

Lemma 3.4. *Let L be a Lie algebra and let $I_1,\ldots,I_n$ be Lie ideals of L. Let $(i_1,i_2,\ldots,i_p)$ be a sequence of integers with $1\leq i_s\leq n$. Suppose that*

$$\{i_1,i_2,\ldots,i_p\}=\{1,2,\ldots,n\}.$$

Then

$$[[I_{i_1},I_{i_2}],\ldots,I_{i_p}]\leq[[I_1,I_2],\ldots,I_n]_S.$$

Proof. We also apply double induction. The first induction is on n. The assertion clearly holds for $n=1$. Suppose that the assertion holds for $n-1$ with $n>1$. From the condition $\{i_1,i_2,\ldots,i_p\}=\{1,2,\ldots,n\}$, we have $p\geq n$. When $p=n$, $(i_1,\ldots,i_n)$ is a permutation of $(1,\ldots,n)$ and so

$$[[I_{i_1},I_{i_2}],\ldots,I_{i_n}]\leq[[I_1,I_2],\ldots,I_n]_S.$$

Suppose that

$$[[I_{j_1},I_{j_2}],\ldots,I_{j_q}]\leq[[I_1,I_2],\ldots,I_n]_S$$

for any sequence $(j_1, \ldots, j_q)$ with $q < p$ and $\{j_1, \ldots, j_q\} = \{1, \ldots, n\}$. Let $(i_1, \ldots, i_p)$ be a sequence with $\{i_1, \ldots, i_p\} = \{1, \ldots, n\}$. If $i_p \in \{i_1, \ldots, i_{p-1}\}$, then $\{i_1, \ldots, i_{p-1}\} = \{1, \ldots, n\}$ and so

$$[[I_{i_1}, I_{i_2}], \ldots, I_{i_{p-1}}] \leq [[I_1, I_2], \ldots, I_n]_S$$

by the second induction hypothesis. It follows that

$$[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}] \leq [[I_1, I_2], \ldots, I_n]_S.$$

If $i_p \notin \{i_1, \ldots, i_{p-1}\}$, we may assume that $i_p = n$. Then

$$\{i_1, \ldots, i_{p-1}\} = \{1, \ldots, n-1\}$$

and so

$$[[I_{i_1}, I_{i_2}], \ldots, I_{i_{p-1}}] \leq [[I_1, I_2], \ldots, I_{n-1}]_S$$

by the first induction hypothesis. From Lemma 3.3, we have

$$[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}] \leq [[I_1, I_2], \ldots, I_n]_S.$$

The inductions are finished, hence the result holds. □

Lemma 3.5. *Let L be a Lie algebra and let $I_1, \ldots, I_n$ be Lie ideal of L with $n \geq 2$. Let $(i_1, \ldots, i_p)$ and $(j_1, \ldots, j_q)$ be sequences of integers such that $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2, \ldots, n\}$. Then*

$$[[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}], [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}]] \leq [[I_1, I_2], \ldots, I_n]_S.$$

Proof. Again we use the double induction on n and q with $n \geq 2$ and $q \geq 1$. First we prove that the assertion holds for $n = 2$. If $\{i_1, \ldots, i_p\} = \{1, 2\}$ or $\{j_1, \ldots, j_q\} = \{1, 2\}$, we have

$$[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}] \leq [[I_1, I_2]_S \text{ or } [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}] \leq [[I_1, I_2]_S$$

by Lemma 3.4 and so

$$[[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}], [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}]] \leq [[I_1, I_2]_S.$$

Otherwise, $i_1 = \cdots = i_p$ and $j_1 = \cdots = j_q$, since $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2\}$, we may assume that $i_1 = \cdots = i_p = 1, j_1 = \cdots = j_q = 2$, then

$$[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}] \leq I_1 \text{ and } [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}] \leq I_2$$

and so

$$[[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}], [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}]] \leq [[I_1, I_2]_S.$$

Suppose the assertion holds for $n-1$, that is

$$[[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}], [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}]] \leq [[I_1, I_2], \ldots, I_{n-1}]_S.$$

when $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2, \ldots, n-1\}$. We shall use the second induction on q to prove that the assertion holds for n. If $q = 1$, the assertion follows by Lemma 3.4. Suppose that the assertion holds

for $q-1$. By Lemma 3.2, $[[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}], [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}]]$ is a Lie ideal of the sum

$$[[[[I_{i_1}, \ldots, I_{i_p}], [[I_{j_1}, \ldots, I_{j_{q-1}}]], I_{j_q}] + [[[[I_{i_1}, \ldots, I_{i_p}], I_{j_q}], [[I_{j_1}, \ldots, I_{j_{q-1}}]].$$

By the second induction we have

$$[[[[I_{i_1}, \ldots, I_{i_p}], I_{j_q}], [[I_{j_1}, \ldots, I_{j_{q-1}}]] \leq [[I_1, I_2], \ldots, I_n]_S.$$

If $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} = \{1, 2, \ldots, n\}$, by the second induction

$$[[[I_{i_1}, \ldots, I_{i_p}], [[I_{j_1}, \ldots, I_{j_{q-1}}]] \leq [[I_1, I_2], \ldots, I_n]_S$$

and hence

$$[[[[I_{i_1}, \ldots, I_{i_p}], [[I_{j_1}, \ldots, I_{j_{q-1}}]], I_{j_q}] \leq [[I_1, I_2], \ldots, I_n]_S.$$

If $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} \neq \{1, 2, \ldots, n\}$, we may assume that

$$\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} = \{1, 2, \ldots, n-1\}$$

and $j_q = n$. By the first induction,

$$[[[I_{i_1}, \ldots, I_{i_p}], [[I_{j_1}, \ldots, I_{j_{q-1}}]] \leq [[I_1, I_2], \ldots, I_{n-1}]_S.$$

Then

$$[[[[I_{i_1}, \ldots, I_{i_p}], [[I_{j_1}, \ldots, I_{j_{q-1}}]], I_{j_q}] \leq [[I_1, I_2], \ldots, I_n]_S.$$

It follows that $[[[I_{i_1}, I_{i_2}], \ldots, I_{i_p}], [[I_{j_1}, I_{j_2}], \ldots, I_{j_q}]] \leq [[I_1, I_2], \ldots, I_n]_S$. The double induction is finished, hence the result. □

Proof of Theorem 3.1. Clearly $[[I_1, I_2], \ldots, I_n]_S \leq [[I_1, I_2, \ldots, I_n]]$. We prove by induction on n that

$$[[I_1, I_2, \ldots, I_n]] \leq [[I_1, I_2], \ldots, I_n]_S.$$

The assertion holds for $n = 1$. We now make the first induction hypothesi that for all $1 \leq s < n$ and for any Lie ideals $I_1, \ldots, I_s$ of L

$$[[I_1, I_2, \ldots, I_s]] \leq [[I_1, I_2], \ldots, I_s]_S. \tag{3.2}$$

Let $I_1, \ldots, I_n$ be arbitrary Lie ideals of L. By definition, $[[I_1, I_2, \ldots, I_n]]$ is generated by all commutators

$$\beta^t(a_{i_1}, \ldots, a_{i_t})$$

of weight t such that $\{i_1, i_2, \ldots, i_t\} = \{1, 2, \ldots, n\}$ with $a_j \in I_j$. To prove that each generator $\beta^t(a_{i_1}, \ldots, a_{i_t}) \in [[I_1, I_2], \ldots, I_n]_S$, we start the second induction on the weight t of β^t with $t \geq n$. If $t = n$, then $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$ and so the assertion holds by Lemma 3.3. Let $n \leq k$ and let

$$\beta^k(a'_{i_1}, \ldots, a'_{i_k})$$

be any bracket arrangement of weight k such that

1) $1 \le i_s \le n$;
2) $\{i_1, \dots, i_k\} = \{1, \dots, n\}$;
3) $a'_j \in I_j$;

Now assume that the second hypothesis holds:

$$\beta^k(a'_{i_1}, \dots, a'_{i_k}) \in [[I_1, I_2], \dots, I_n]_S \tag{3.3}$$

for all k such that $n \le k < t$.

Let $\beta^t(a_{i_1}, \dots, a_{i_t})$ be any bracket arrangement of weight t with $\{i_1, \dots, i_t\} = \{1, \dots, n\}$ and $a_j \in I_j$ for $1 \le j \le n$. From the definition of bracket arrangement, we have

$$\beta^t(a_{i_1}, \dots, a_{i_t}) = [\beta^p(a_{i_1}, \dots, a_{i_p}), \beta^{t-p}(a_{i_{p+1}}, \dots, a_{i_t})]$$

for some bracket arrangements β^p and β^{t-p} of weight p and $t-p$, respectively, with $1 \le p \le n-1$. Let

$$A = \{i_1, \dots, i_p\} \text{ and } B = \{i_{p+1}, \dots, i_t\}.$$

Then both A and B are the subsets of $\{1, \dots, n\}$ with $A \cup B = \{1, \dots, n\}$.

Suppose that the cardinality $|A| = n$ or $|B| = n$. We may assume that $|A| = n$. By hypothesis 3.3,

$$\beta^p(a_{i_1}, \dots, a_{i_p}) \in [[I_1, I_2], \dots, I_n]_S.$$

Since $[[I_1, I_2], \dots, I_n]_S$ is a Lie ideal of L, we have

$$\beta^t(a_{i_1}, \dots, a_{i_t}) = [\beta^p(a_{i_1}, \dots, a_{i_p}), \beta^{t-p}(a_{i_{p+1}}, \dots, a_{i_t})] \in [[I_1, I_2], \dots, I_n]_S.$$

This proves the result in this case.

Suppose that $|A| < n$ and $|B| < n$. Let $A = \{l_1, \dots, l_a\}$ with $1 \le l_1 < l_2 < \cdots < l_a \le n$ and $1 \le a < n$, and let $B = \{k_1, \dots, k_b\}$ with $1 \le k_1 < k_2 < \cdots < k_b$ and $1 \le b < n$. Observe that

$$\beta^p(a_{i_1}, \dots, a_{i_p}) \in [[I_{l_1}, I_{l_2}, \dots, I_{l_a}]].$$

By hypothesis 3.2,

$$[[I_{l_1}, I_{l_2}, \dots, I_{l_a}]] = [[I_{l_1}, I_{l_2}], \dots, I_{l_a}]_S.$$

Thus

$$\beta^p(a_{i_1}, \dots, a_{i_p}) \in [[I_{l_1}, I_{l_2}], \dots, I_{l_a}]_S.$$

Similarly

$$\beta^{t-p}(a_{i_{p+1}}, \dots, a_{i_t}) \in [[I_{k_1}, I_{k_2}], \dots, I_{k_b}]_S.$$

It follows that

$$\beta^t(a_{i_1}, \dots, a_{i_t}) \in [[[I_{l_1}, I_{l_2}], \dots, I_{l_a}]_S, [[I_{k_1}, I_{k_2}], \dots, I_{k_b}]_S].$$

From Lemma 3.5, we have

$$\left[[[I_{l_{\sigma(1)}}, I_{l_{\sigma(2)}}], \dots, I_{l_{\sigma(a)}}], [[I_{k_{\tau(1)}}, I_{k_{\tau(2)}}], \dots, I_{k_{\tau(b)}}]\right] \le [[I_1, I_2], \dots, I_n]_S$$

for all $\sigma \in \Sigma_a$ and $\tau \in \Sigma_b$ because $\{l_1, \ldots, l_a\} \cup \{k_1, \ldots, k_b\} = A \cup B = \{1, 2, \ldots, n\}$. It follows that

$$[[[I_{l_1}, I_{l_2}], \ldots, I_{l_a}]_S, [[I_{k_1}, I_{k_2}], \ldots, I_{k_b}]_S] \le [[I_1, I_2], \ldots, I_n]_S.$$

Thus

$$\beta^t(a_{i_1}, \ldots, a_{i_t}) \in [[I_1, I_2], \ldots, I_n]_S.$$

The inductions are finished, hence Theorem 3.1.

Let us denote the ideal

$$L[A_{k,n}, [\cdots[A_{k,n}, A_{j_1,n}], \ldots, A_{j_m,n}] \mid j_i \neq k, n;\ j_i \le n-2, i \le m;\ m \ge 1]$$

by I_k. Then we have the following theorem.

Theorem 3.6. *The Lie subalgebra* $\mathrm{L^P(Brun_n)}$ *and the symmetric bracket sum* $[[I_1, I_2], \ldots, I_{n-1}]_S$ *are equal as subalgebras in* $L(P_n)$*:*

$$\mathrm{L^P(Brun_n)} = [[I_1, I_2], \ldots, I_{n-1}]_S.$$

Proof. It is evident that the symmetric bracket sum $[[I_1, I_2], \ldots, I_{n-1}]_S$ lies in the kernels of all d_i. On the other hand, from Theorem 2.11, $\mathrm{L^P(Brun_n)}$ is given as "*fat Lie product*" of $I_1, \ldots, I_{n-1}$ because each element in $\mathcal{K}(n)_1$ is a Lie monomial containing each of $A_{1,n}, \ldots, A_{n-1,n}$. We know that $\mathcal{K}(n)_1 \subseteq [[I_1, \ldots, I_{n-1}]] = [[I_1, I_2], \ldots, I_{n-1}]_S$. Thus $\mathrm{L^P(Brun_n)}$ is contained in the symmetric bracket sum $[[I_1, I_2], \ldots, I_{n-1}]_S$. □

4. The Rank of $L_q^P(\mathrm{Brun}_n)$

Observe that the Lie algebra $L(P)$ is of finite type in the sense that each homogeneous component $L_k(P_n)$ is a free abelian group of finite rank. Thus the subgroup

$$\mathrm{L^P(Brun_n)} \cap L_k(P_n)$$

is a free abelian group of finite rank. The purpose of this section to give a formula on the rank of $L_q^P(\mathrm{Brun}_n)$

4.1. **A decomposition formula on bi-Δ-groups.** By the definition of bi-Δ-groups and the face and co-face operation on $\mathbb{P} = \{\mathbb{P}_n\}_{n\ge 0}$, we have the following lemma.

Lemma 4.1. *For every* $q \ge 0$, $L_q(\mathbb{P}) = \{L_q(\mathbb{P}_n)\}_{n\ge 0}$ *is a bi-Δ-group.* □

Let $\mathcal{G} = \{G_n\}_{n\ge 0}$ be a bi-Δ-group. Define

$$\mathcal{Z}_n(\mathcal{G}) = \bigcap_{i=0}^{n} \mathrm{Ker}(d_i \colon G_n \to G_{n-1}).$$

The following statement on bi-Δ-groups is proved in [23, Proposition 1.2.9].

Theorem 4.2 (Decomposition Theorem of bi-Δ-groups)**.** *Let $\mathcal{G} = \{G_n\}_{n\geq 0}$ be a bi-Δ-group. Then G_n is the (iterated) semi-direct product the subgroups*

$$d^{i_k}d^{i_{k-1}}\cdots d^{i_1}(\mathcal{Z}_{n-k}(\mathcal{G})),$$

$0\leq i_1<\cdots<i_k\leq n$, $0\leq k\leq n$, with lexicographic from right. $\square$

Corollary 4.3. *Let $\mathcal{G} = \{G_n\}_{n\geq 0}$ be a bi-Δ-group such that each G_n is an abelian group. Then there is direct sum decomposition*

$$G_n = \bigoplus_{\substack{0\leq i_1<\cdots<i_k\leq n\\ 0\leq k\leq n}} d^{i_k}d^{i_{k-1}}\cdots d^{i_1}(\mathcal{Z}_{n-k}(\mathcal{G}))$$

for each n. $\square$

4.2. **The Rank of $L^P_q(\mathrm{Brun}_n)$.** Let $\mathcal{G} = L_q(\mathbb{P})$. Then $\mathcal{Z}_n(L_q(\mathbb{P}) = L^P_q(\mathrm{Brun}_{n+1})$ by Proposition 2.3. Let $d^i = \mathbb{d}^{i-1}: \mathbb{P}_{n-1} = P_n \to \mathbb{P}_n = P_{n+1}$ is obtained by adding a trivial ist string in front of the other strings ($i = 1, 2, \cdots, n$). By Corollary 4.3, we have the following decomposition.

Proposition 4.4. *There is a decomposition*

$$L_q(P_n) = \bigoplus_{\substack{1\leq i_1<\cdots<i_k\leq n\\ 0\leq k\leq n-1}} d^{i_k}d^{i_{k-1}}\cdots d^{i_1}(L^P_q(\mathrm{Brun}_{n-k}))$$

for each n and q. $\square$

Corollary 4.5. *There is a formula*

$$\mathrm{rank}(L_q(P_n)) = \sum_{k=0}^{n-1}\binom{n}{k}\mathrm{rank}(L^P_q(\mathrm{Brun}_{n-k}))$$

for each n and q. $\square$

Theorem 4.6.

$$\mathrm{rank}(L^P_q(\mathrm{Brun}_n)) = \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\mathrm{rank}(L_q(P_{n-k}))$$

for each n and q, where $P_1 = 0$ and, for $m\geq 2$,

$$\mathrm{rank}(L_q(P_m)) = \frac{1}{q}\sum_{k=1}^{m-1}\sum_{d|q}\mu(d)k^{q/d}$$

with μ the Möbis function.

Proof. From the semi-direct product decomposition of Lie algebras,

$$L(P_m) \cong L(P_{m-1}) \oplus L(F_{m-1}),$$

we have

$$\operatorname{rank}(L_q(P_m)) = \sum_{k=1}^{m-1} \operatorname{rank}(L_q(F_k))$$

for $m \geq 2$. Since $L(F_k)$ is the free Lie algebra on a set of k-elements,

$$\operatorname{rank}(L_q(F_k)) = \frac{1}{q} \sum_{d|q} \mu(d) k^{q/d}$$

and so

$$\operatorname{rank}(L_q(P_m)) = \frac{1}{q} \sum_{k=1}^{m-1} \sum_{d|q} \mu(d) k^{q/d}.$$

Now let $b_q(P_n) = \operatorname{rank}(L_q(P_n))$ and $b_q^P(\mathrm{Brun}_n) = \operatorname{rank}(L_q^P(\mathrm{Brun}_{n-k}))$. By Corollary 4.5, we have

$$\begin{pmatrix} b_q(P_n) \\ b_q(P_{n-1}) \\ b_q(P_{n-2}) \\ \vdots \\ b_q(P_1) \end{pmatrix} = \begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ 0 & 1 & \binom{n-1}{1} & \cdots & \binom{n-1}{n-2} \\ 0 & 0 & 1 & \cdots & \binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_q^P(\mathrm{Brun}_n) \\ b_q^P(\mathrm{Brun}_{n-1}) \\ b_q^P(\mathrm{Brun}_{n-2}) \\ \vdots \\ b_q^P(\mathrm{Brun}_1) \end{pmatrix}.$$

Let A_n be the coefficient matrix of the above linear equations. Then

$$A_n^{-1} = \begin{pmatrix} 1 & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \cdots & (-1)^{n-1}\binom{n}{n-1} \\ 0 & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \cdots & (-1)^{n-2}\binom{n-1}{n-2} \\ 0 & 0 & 1 & -\binom{n-2}{1} & \cdots & (-1)^{n-3}\binom{n-2}{n-3} \\ 0 & 0 & 0 & 1 & \cdots & (-1)^{n-4}\binom{n-3}{n-4} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and hence the result. $\square$

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